

EULER SEQUENCE FOR COMPLETE SMOOTH \mathbb{K}^* -SURFACES

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ABSTRACT. In this note we introduce exact sequences of sheaves on a complete smooth \mathbb{K}^* -surface without elliptic points. These sequences are an attempt to generalize the Euler sequence for a toric variety to complexity one surfaces. As an application we show that such a surface is rigid if and only if it is Fano.

INTRODUCTION

In [Har77, V.8.13], we are presented with an exact sequence, namely

$$0 \longrightarrow \Omega_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0,$$

which allows the author to make several computations regarding differentials. This sequence is named *Euler sequence* and in [CLS11] it is generalized to any smooth toric variety X coming from a fan Σ whose rays span the whole ambient lattice. The exact sequence in question is

$$0 \longrightarrow \Omega_X \longrightarrow \bigoplus_{\rho \in \Sigma(1)} \mathcal{O}_{\mathbb{P}^n}(-D_\rho) \longrightarrow \text{Pic}(X) \otimes_{\mathbb{Z}} \mathcal{O}_X \longrightarrow 0,$$

where D_ρ is the invariant divisor associated to the ray ρ .

This work attempts to obtain a similar result in the case of \mathbb{K}^* -surfaces.

We briefly recall that a \mathbb{K}^* -surface X without elliptic points (see Definition 1.8) is equipped with an equivariant morphism $\pi: X \rightarrow Y$, onto a smooth projective curve Y , which admits two distinguished sections called the source F^+ and the sink F^- of X . This allows us to form the divisor $E_S = mF - \sum_{i \in I} E_i$, where m is the number of reducible fibers of π and $\{E_i : i \in I\}$ is the set of the prime components of such fibers.

Our main theorem is the following.

Theorem 1. *Let X be a complete smooth \mathbb{K}^* -surface without elliptic points. There is an exact sequence of \mathcal{O}_X -modules*

$$0 \longrightarrow \pi^* \Omega_Y \otimes_{\mathcal{O}_X} \mathcal{O}_X(E_S) \longrightarrow \Omega_X \longrightarrow \mathcal{G} \longrightarrow \mathcal{O}_X^{r+1} \longrightarrow 0$$

where $r = |I| - m$ and \mathcal{G} is the quotient of $\bigoplus_{i \in I} \mathcal{O}_X(E_i) \oplus \mathcal{O}_X(F^+) \oplus \mathcal{O}_X(F^-)$ by the subsheaf $\bigoplus_{i,j} \mathcal{O}_X(-E_i - E_j)$, where the sum is taken over all the pairs (i, j) such that $E_i \cap E_j \neq \emptyset$.

The paper is organized as follows. In Section 1 we introduce the language of polyhedral divisors detailed in [AH06]. Section 2 is devoted to the construction of two exact sequences needed to prove Theorem 1. Because of the method employed to prove them, we restrict our attention to complete smooth \mathbb{K}^* -surfaces without elliptic points. Finally, in Section 3, as an application, we show that the only rigid

rational \mathbb{K}^* -surfaces without elliptic points are Fano. This is a partial generalization of [Ilt11, Corollary 2.8] where the case of toric surfaces is considered.

Theorem 2. *Let X be a complete smooth \mathbb{K}^* -surface without elliptic points. Then the following are equivalent:*

- (1) *The equality $h^1(X, T_X) = 0$ holds;*
- (2) *X is toric Fano.*

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1. PRELIMINARIES

Let \mathbb{K} be an algebraically closed field of characteristic zero and X an algebraic variety over \mathbb{K} having an action of $T := (\mathbb{K}^*)^n$ (which is called the n -dimensional torus). This action is called *effective* if the only $t \in T$, for which $t \cdot x = x$ holds for all $x \in X$, is the identity of T .

Definition. A T -variety is a normal algebraic variety X coming with an effective $(\mathbb{K}^*)^n$ -action. The *complexity* of X is the difference $\dim X - n$.

These varieties admit a polyhedral description given by K. Altmann and J. Hausen for the affine case in [AH06] and later, together with H. Suß, in [AHS08] for the non-affine case. In the following section, we briefly recall this construction as well as a definition by N. Ilten and H. Suß in [IS11] that helps to simplify the notation.

1.1. Polyhedral divisors. Let N be a lattice of rank n , and $M = \text{Hom}(N, \mathbb{Z})$ its dual. We denote by $\langle -, - \rangle : M \times N \rightarrow \mathbb{Z}$ the perfect pairing defined by $(u, v) \mapsto \langle u, v \rangle := u(v)$ and by $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$, $M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$ the rational vector spaces. A *polyhedron* in $N_{\mathbb{Q}}$ is an intersection of finitely many affine half spaces in $N_{\mathbb{Q}}$. If we require the supporting hyperplane of any half space to be a linear subspace, the polyhedron is called a *cone*. If σ is a cone in $N_{\mathbb{Q}}$, its *dual cone* is defined as

$$\sigma^{\vee} := \{u \in M_{\mathbb{Q}} : \langle u, v \rangle \geq 0 \text{ for all } v \in N_{\mathbb{Q}}\}.$$

Let $\Delta \subseteq N_{\mathbb{Q}}$ be a polyhedron. The set

$$\sigma := \{v \in N_{\mathbb{Q}} : tv + \Delta \subseteq \Delta, \forall t \in \mathbb{Q}\}$$

is a cone called the *tailcone* of Δ and Δ is called a σ -polyhedron.

Definition 1.1. Let Y be a normal variety and σ a cone. A *polyhedral divisor* on Y is a formal sum

$$\mathcal{D} := \sum_P \Delta_P \otimes P,$$

where P runs over all prime divisors of Y and the Δ_P are all σ -polyhedrons such that $\Delta_P = \sigma$ for all but finitely many P . We admit the empty set as a valid σ -polyhedron too.

Let $\mathfrak{D} := \sum \Delta_P \otimes P$ be a polyhedral divisor on Y , with tailcone σ . For every $u \in \sigma^{\vee}$ we define the evaluation

$$\mathfrak{D}(u) := \sum_{\substack{P \subset Y \\ \Delta_P \neq \emptyset}} \min_{v \in \Delta_P} \langle u, v \rangle \otimes P \in \text{WDiv}_{\mathbb{Q}}(\text{Loc } \mathcal{D})$$

where $\text{Loc } \mathcal{D} := Y \setminus (\cup_{\Delta_P = \emptyset} P)$ is the *locus* of \mathcal{D} .

Definition 1.2. Let Y be a normal variety. A *proper polyhedral divisor*, also called a *pp-divisor* is a polyhedral divisor \mathcal{D} on Y , such that

- (i) $\mathcal{D}(u)$ is Cartier and semiample for every $u \in \sigma^\vee \cap M$.
- (ii) $\mathcal{D}(u)$ is big for every $u \in (\text{relint } \sigma^\vee) \cap M$.

Now, let \mathcal{D} be a pp-divisor on a semiprojective (i.e. projective over some affine variety) variety Y , \mathcal{D} having tailcone $\sigma \subseteq N_{\mathbb{Q}}$. This defines an M -graded algebra

$$A(\mathcal{D}) := \bigoplus_{u \in \sigma^\vee \cap M} \Gamma(\text{Loc } \mathcal{D}, \mathcal{O}(\mathcal{D}(u))).$$

The affine scheme $X(\mathcal{D}) := \text{Spec } A(\mathcal{D})$ comes with a natural action of $\text{Spec } \mathbb{K}[M]$. Definition 1.2 is mainly motivated by the following result [AH06, Theorem 3.1 and Theorem 3.4].

Theorem 1.3. *Let \mathcal{D} be a pp-divisor on a normal variety Y . Then $X(\mathcal{D})$ is an affine T -variety of complexity equal to $\dim Y$. Moreover, every affine T -variety arises like this.*

1.2. Divisorial fans. Non-affine T -varieties are obtained by gluing affine T -varieties coming from pp-divisors in a combinatorial way as specified in Definition 1.4.

Consider two polyhedral divisors $\mathcal{D} = \sum \Delta_P \otimes P$ and $\mathcal{D}' = \sum \Delta'_P \otimes P$ on Y , with tailcones σ and σ' respectively and such that $\Delta_P \subseteq \Delta'_P$ for every P . We then have an inclusion

$$\bigoplus_{u \in \sigma^\vee \cap M} \Gamma(\text{Loc } \mathcal{D}, \mathcal{O}(\mathcal{D}(u))) \subseteq \bigoplus_{u \in \sigma'^\vee \cap M} \Gamma(\text{Loc } \mathcal{D}, \mathcal{O}(\mathcal{D}'(u))),$$

which induces a morphism $X(\mathcal{D}') \rightarrow X(\mathcal{D})$. We say that \mathcal{D}' is a *face* of \mathcal{D} , denoted by $\mathcal{D}' \prec \mathcal{D}$, if this morphism is an open embedding.

Definition 1.4. A *divisorial fan* on Y is a finite set \mathcal{S} of pp-divisors on Y such that for every pair of divisors $\mathcal{D} = \sum \Delta_P \otimes P$ and $\mathcal{D}' = \sum \Delta'_P \otimes P$ in \mathcal{S} , we have $\mathcal{D} \cap \mathcal{D}' \in \mathcal{S}$ and $\mathcal{D} \succ \mathcal{D} \cap \mathcal{D}' \prec \mathcal{D}'$, where $\mathcal{D} \cap \mathcal{D}' := \sum (\Delta_P \cap \Delta'_P) \otimes P$.

This definition allows us to glue affine T -varieties via

$$X(\mathcal{D}) \longleftarrow X(\mathcal{D} \cap \mathcal{D}') \longrightarrow X(\mathcal{D}'),$$

thus resulting in a scheme $X(\mathcal{S})$. For the following theorem see [AHS08, Theorem 5.3 and Theorem 5.6].

Theorem 1.5. *The scheme $X(\mathcal{S})$ constructed above is a T -variety of complexity equal to $\dim Y$. Every T -variety can be constructed like this.*

1.3. Marked fansy divisors on curves. In this subsection Y will be a smooth projective curve. For a polyhedral divisor $\mathcal{D} = \sum_P \Delta_P \otimes P$ on Y , and a point $y \in Y$, set

$$\mathcal{D}_y := \sum_{P \ni y} \Delta_P.$$

Then for a divisorial fan \mathcal{S} on Y , we define the *slice* of \mathcal{S} at y as $\{\mathcal{D}_y : \mathcal{D} \in \mathcal{S}\}$.

Now, the slices of a divisorial fan do not give enough information about the corresponding T -variety. Two divisorial fans \mathcal{S} and \mathcal{S}' can have the same slices, yet $X(\mathcal{S}) \neq X(\mathcal{S}')$. In [IS11], this issue is fixed for the case of complete complexity-one T -varieties with the following definition.

Definition 1.6. A *marked fancy divisor* on a curve Y is a formal sum $\Xi = \sum_{P \in Y} \Xi_P \otimes P$ together with a fan Σ and a subset $C \subseteq \Sigma$, such that

- (i) Ξ_P is a complete polyhedral subdivision of $N_{\mathbb{Q}}$, and $\text{tail}(\Xi_P) = \Sigma$ for all $P \in Y$.
- (ii) For $\sigma \in C$ of full dimension, $\sum \Delta_P^\sigma \otimes P$ is a pp-divisor, where Δ_P^σ is the only σ -polyhedron of Ξ_P .
- (iii) For $\sigma \in C$ of full dimension and $\tau \prec \sigma$, we have $\tau \in C$ if and only if $(\sum_P \Delta_P^\sigma) \cap \tau \neq \emptyset$.
- (iv) If $\tau \prec \sigma$ and $\tau \in C$, then $\sigma \in C$.

We say that the cones in C are *marked*.

Given a complete divisorial fan \mathcal{S} on a curve Y , we can define the marked fancy divisor $\Xi = \sum_P \mathcal{S}_P \otimes P$ with marks on all the tailcones of divisors $\mathcal{D} \in \mathcal{S}$ having complete locus. We denote it by $\Xi(\mathcal{S})$. The following is proved in [IS11, Proposition 1.6].

Proposition 1.7. *For any marked fancy divisor Ξ , there exists a complete divisorial fan \mathcal{S} with $\Xi(\mathcal{S}) = \Xi$. If two divisorial fans $\mathcal{S}, \mathcal{S}'$ satisfy $\Xi(\mathcal{S}) = \Xi(\mathcal{S}')$, then $X(\mathcal{S}) = X(\mathcal{S}')$.*

1.4. \mathbb{K}^* -surfaces. We now look at the case of a T -variety of dimension two and complexity one. We call this type of variety a \mathbb{K}^* -surface.

Definition 1.8. Let X be a \mathbb{K}^* -surface and let $x \in X$ be a fixed point for the torus action. We say that the fixed point x is:

- *elliptic* if there is an invariant open neighborhood U of x such that this point lies in the closure of every orbit of U ,
- *parabolic* if x lies on a curve made entirely of fixed points,
- *hyperbolic* otherwise.

Before proceeding recall that a morphism of varieties $\pi: X \rightarrow Y$ where X is a T -variety is called a *good quotient* if π is affine, constant on the orbits and the pullback $\pi^*: \mathcal{O}_Y \rightarrow \pi_*(\mathcal{O}_X)^T$ is an isomorphism.

Proposition 1.9. *Let X be a complete \mathbb{K}^* -surface. The following statements are equivalent.*

- (i) *There exists a morphism $X \rightarrow Y$ onto a smooth projective curve Y that is a good quotient for the \mathbb{K}^* -action.*
- (ii) *X has no elliptic fixed points.*
- (iii) *X is given by a marked fancy divisor without marks.*

Proof. We prove three implications.

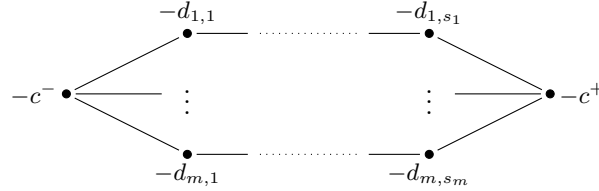
(i) \Rightarrow (ii). Let $x \in X$ be an elliptic fixed point. There is an open neighborhood U of x such that x lies in the closure of every orbit in U . Therefore, there cannot be a good quotient $X \rightarrow Y$ because the open set U would be mapped to a single point; a contradiction.

(ii) \Rightarrow (iii). Assume that the marked fancy divisor defining X has a mark. There is then an open affine chart of X given by a pp-divisor \mathcal{D} with complete locus Y . The zero-degree component of $A(\mathcal{D})$ is $\Gamma(Y, \mathcal{O}) \cong \mathbb{K}$, meaning that the degrees of the generators of $A(\mathcal{D})$ as an algebra are either all positive (if $\text{tail } \mathcal{D} = \mathbb{Q}_{\geq 0}$) or all negative (if $\text{tail } \mathcal{D} = \mathbb{Q}_{\leq 0}$). In either case, we can take local coordinates such that the origin x_0 lies in the closure of every orbit, i.e. x_0 is an elliptic fixed point.

(iii) \Rightarrow (i). Let \mathcal{S} be a divisorial fan on a smooth projective curve Y such that the marked fansy divisor for X is $\Xi(\mathcal{S})$. Since there are no marks, each $\mathcal{D} \in \mathcal{S}$ has an affine locus, so there is a morphism $X(\mathcal{D}) \rightarrow \text{Loc } \mathcal{D}$ coming from the inclusion $A(\mathcal{D})_0 := \Gamma(\text{Loc } \mathcal{D}, \mathcal{O}) \subseteq A(\mathcal{D})$. These glue together to a morphism $\pi: X \rightarrow Y$ because the completeness of X implies that $\{\text{Loc } \mathcal{D} : \mathcal{D} \in \mathcal{S}\}$ is an affine open covering of Y . Thus π is a good quotient because $A(\mathcal{D})_0$ is precisely the subalgebra of invariants of $A(\mathcal{D})$. \square

Let X be a complete \mathbb{K}^* -surface. There exist two invariant subsets $F^- \subseteq X$ and $F^+ \subseteq X$, called *sink* and *source* respectively, such that there is an open set $U \subseteq X$ where the closure in X of every orbit in U intersects both F^- and F^+ . There are finitely many orbits outside of $U \cup F^- \cup F^+$, that are called the *special* orbits. The source can be either an elliptic point or an irreducible curve of parabolic points; the same holds true for the sink. Every fixed point outside of $F^+ \cup F^-$ is hyperbolic.

Now, consider a complete smooth \mathbb{K}^* -surface having no elliptic points. Denote by E_1, \dots, E_r the closures of the special orbits. F. Orlik and P. Wagreich construct a graph having vertex set $\{E_1, \dots, E_r, F^+, F^-\}$ and two vertices are joined by an edge if and only if the two corresponding curves intersect. Each vertex carries a weight equal to the self-intersection number of the curve that it represents. This graph takes the following form.

FIGURE 1. The graph of the \mathbb{K}^* -surface

The $d_{i,j}$ are all positive and satisfy that the Hirzebruch-Jung continued fraction $[d_{i,1}, d_{i,2}, \dots, d_{i,s_i}]$ equals 0 for every $1 \leq i \leq m$. On the other hand, our surface is given by a marked fansy divisor, without marks, on a smooth projective curve Y .

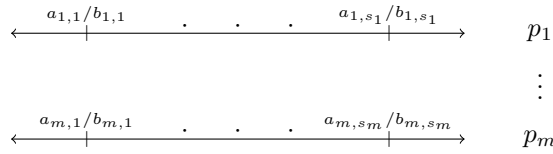


FIGURE 2. Marked fansy divisor without marks

Where the smoothness of the surface implies that $b_{i,j}a_{i,j+1} - a_{i,j}b_{i,j+1} = 1$ for every $1 \leq i \leq m$, $1 \leq j < s_i$, as well as $b_{i,1} = b_{i,s_i} = 1$, as shown in [Süß10, Theorem 3.3]. It turns out, as well, that each $b_{i,j}$ with $j > 1$ equals the Hirzebruch-Jung continued fraction $[d_{i,1}, d_{i,2}, \dots, d_{i,j-1}]$.

2. EULER SEQUENCE

2.1. Euler sequence for \mathbb{K}^* -surfaces. Let X be a complete smooth \mathbb{K}^* -surface having no elliptic points given by a marked fancy divisor Ξ with no marks, like the one depicted in Figure 2. Each fraction $a_{i,j}/b_{i,j}$ in the figure corresponds to a divisor $E_{i,j}$ which is the closure of a special orbit of X .

Definition 2.1. The multiplicity of the divisor $E_{i,j}$ is the non-negative integer

$$\mu(E_{i,j}) := b_{i,j} - 1.$$

According to the definition of the divisor $E_{\mathcal{S}}$ given in the introduction the equality $E_{\mathcal{S}} := \sum_{i,j} \mu(E_{i,j}) \cdot E_{i,j}$ holds, where $1 \leq i \leq m$ and $1 \leq j \leq s_i$.

Let Ω_X and Ω_Y be the cotangent sheaves of X and Y respectively. As in Section 1.4, let F^- and F^+ denote the source and sink of X . Let $Z \subseteq X$ be the set of hyperbolic fixed points of X . In what follows we will call \mathcal{F}_X the sheaf $\mathcal{O}_{F^-} \oplus \mathcal{O}_{F^+} \oplus \mathcal{O}_Z$.

Lemma 2.2. *Let X be a complete smooth \mathbb{K}^* -surface without elliptic points. There is an exact sequence of \mathcal{O}_X -modules*

$$0 \longrightarrow \pi^* \Omega_Y \otimes_{\mathcal{O}_X} \mathcal{O}_X(E_{\mathcal{S}}) \xrightarrow{\iota} \Omega_X \xrightarrow{\alpha} \mathcal{O}_X \longrightarrow \mathcal{F}_X \longrightarrow 0$$

where α is defined by $f dz \mapsto \deg(z) f z$, for any homogeneous local coordinate z with respect to the \mathbb{Z} -grading of \mathcal{O}_X induced by the \mathbb{K}^* -action and ι is defined by $dt \otimes f \mapsto f dt$, where t is the pull-back of a local coordinate on Y .

Proof. Let Ξ be a fancy divisor describing X . Each affine chart of X , or an intersection of them, is given by a polyhedron Δ on some slice of Ξ . In other words, it is given by the pp-divisor

$$\mathcal{D} = \Delta \otimes p + \sum_{\mathcal{P} - \{p\}} \emptyset \otimes p,$$

where \mathcal{P} is the set of points of Y with non-trivial slices. We analyze each possible Δ separately.

Case 1. $\Delta = [a_1/b_1, a_2/b_2]$ with $b_1 b_2 \neq 0$. In this case \mathcal{D} defines an open affine subset $X_{\mathcal{D}}$ of X which is the spectrum of the algebra

$$\bigoplus_{u \in \mathbb{Z}} \Gamma(\text{loc}(\mathcal{D}), \mathcal{O}(\mathcal{D}(u))) = S^{-1} \mathbb{K}[x^{a_2} \chi^{-b_2}, x^{-a_1} \chi^{b_1}] \cong S^{-1} \mathbb{K}[z, w] =: R,$$

where x is a regular function of $\text{loc}(\mathcal{D})$ which has a simple zero at p and is non-zero at $\text{loc}(\mathcal{D}) - \{p\}$, while $S \subseteq \mathbb{K}[x^{a_2} \chi^{-b_2}, x^{-a_1} \chi^{b_1}]$ is the multiplicative system defined by degree zero homogeneous polynomials which do not vanish on $\text{loc}(\mathcal{D})$. The first equality is due to our assumption on \mathcal{D} . We have an exact sequence

$$R dz \oplus R dw \cong \Omega_R \xrightarrow{\alpha} R \longrightarrow R/I \longrightarrow 0$$

where $I \subseteq R$ is the ideal $\langle \deg(z)z, \deg(w)w \rangle$. The restriction of the quotient map $\pi: X \rightarrow Y$ to the open subset $X_{\mathcal{D}}$ is defined by the inclusion $\mathbb{K}[x] \subseteq R$. Since $x = z^{\deg(w)} w^{\deg(z)} = z^{b_1} w^{b_2}$, the curve $\pi^{-1}(p) \cap X_{\mathcal{D}}$ has two irreducible components which are vertical curves intersecting at the fixed point $q \in Z$ of local coordinates $z = w = 0$. Thus R/I defines the skyscraper sheaf \mathcal{O}_q and we get

the first exact sequence from $\mathcal{F}|_{X_{\mathcal{D}}} \cong \mathcal{O}_q$. The sheaf $\pi^*\Omega_Y$ is locally generated by $dx = z^{b_1-1}w^{b_2-1}(b_1wdz + b_2zdw)$, thus we have the desired isomorphism

$$\pi^*\Omega_Y \otimes_{\mathcal{O}_X} \mathcal{O}_X((b_1-1)E_1 + (b_2-1)E_2)|_{X_{\mathcal{D}}} \rightarrow \ker(\alpha)|_{X_{\mathcal{D}}},$$

where for $i = 1, 2$, the divisor E_i is the one associated to the fraction a_i/b_i , as explained in the beginning of this section.

Case 2. $\Delta = [a_1, \infty)$. In this case \mathcal{D} defines an open affine subset $X_{\mathcal{D}}$ of X which is the spectrum of the algebra

$$\bigoplus_{u \in \mathbb{Z}_{\geq 0}} \Gamma(\text{loc}(\mathcal{D}), \mathcal{O}(\mathcal{D}(u))) = S^{-1}\mathbb{K}[x, x^{-a_1}\chi] \cong S^{-1}\mathbb{K}[z, w] =: R,$$

where x and S are defined in a similar way as in the first case. Since $x = z$, the curve $\pi^{-1}(p) \cap X_{\mathcal{D}}$ has one irreducible component which is a vertical curve intersecting F^+ at one point. Again we got an exact sequence as above and observe that now $I = \langle w \rangle$. In this case R/I defines the sheaf $\mathcal{O}_{F^+}|_{X_{\mathcal{D}}}$ and we get the first exact sequence from $\mathcal{F}|_{X_{\mathcal{D}}} \cong \mathcal{O}_{F^+}|_{X_{\mathcal{D}}}$. The sheaf $\pi^*\Omega_Y$ is locally generated by $dx = dz$, thus we have an isomorphism

$$\pi^*\Omega_Y|_{X_{\mathcal{D}}} \rightarrow \ker(\alpha)|_{X_{\mathcal{D}}}.$$

Case 3. $\Delta = (-\infty, a_2]$. This is similar to the previous case and we omit the details.

Case 4. $\Delta = \{a/b\}$. In this case \mathcal{D} defines an open affine subset $X_{\mathcal{D}}$ of X which is the spectrum of the algebra

$$\bigoplus_{u \in \mathbb{Z}} \Gamma(\text{loc}(\mathcal{D}), \mathcal{O}(\mathcal{D}(u))) = S^{-1}\mathbb{K}[x^k\chi^{-l}, (x^{-a}\chi^b)^{\pm 1}] \cong S^{-1}\mathbb{K}[z, w^{\pm 1}] =: R,$$

where x and S are defined in a similar way as in the first case and c, d are integers such that $bk - la = 1$. Since $x = z^{\deg(w)}w^{\deg(z)} = z^bw^l$, the curve $\pi^{-1}(p) \cap X_{\mathcal{D}}$ has an irreducible component which is a vertical curve which has empty intersection with $F^+ \cup F^- \cup Z$. Again we get an exact sequence as in the first case with $I = \langle z, w^{\pm 1} \rangle = R$. In this case $R/I = 0$ and we get the first exact sequence from $\mathcal{F}|_{X_{\mathcal{D}}} = 0$. The sheaf $\pi^*\Omega_Y$ is locally generated by $dx = z^{b-1}w^{l-1}(bwdz + lzd w)$, thus we have an isomorphism (since w is a unit in this chart)

$$\pi^*\Omega_Y \otimes_{\mathcal{O}_X} \mathcal{O}_X((b-1)E)|_{X_{\mathcal{D}}} \rightarrow \ker(\alpha)|_{X_{\mathcal{D}}},$$

where E is the divisor associated to the fraction a/b . □

We define the sheaf $\mathcal{Q}_{\alpha} := \Omega_X / \ker(\alpha)$.

Now, maintaining the same hypothesis and notation as above, we are ready to prove part two of Theorem 2.2.

Lemma 2.3. *There is a short exact sequence of \mathcal{O}_X -modules*

$$0 \longrightarrow \mathcal{Q}_{\alpha} \longrightarrow \mathcal{G} \longrightarrow \mathbb{Z}^{r+1} \otimes \mathcal{O}_X \longrightarrow 0,$$

where

$$\mathcal{G} = \mathcal{O}(-F^-) \oplus \mathcal{O}(-F^+) \oplus \left((\oplus_{i,j} \mathcal{O}(-E_{i,j})) / (\oplus_{i=1}^m \oplus_{j=1}^{s_i} \mathcal{O}(-E_{i,j} - E_{i,j+1})) \right).$$

Proof. Let us consider the diagram

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \vdots & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{Q}_\alpha & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{F}_X \longrightarrow 0 \\
& & \vdots & & \downarrow & & \parallel \\
0 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathbb{Z}^{r+2} \otimes \mathcal{O}_X & \longrightarrow & \mathcal{F}_X \longrightarrow 0 \\
& & \vdots & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathbb{Z}^{r+1} \otimes \mathcal{O}_X & \xlongequal{\quad} & \mathbb{Z}^{r+1} \otimes \mathcal{O}_X & \longrightarrow & 0 \\
& & \vdots & & \downarrow & & \\
& & 0 & & 0 & &
\end{array}$$

where the top row comes from Lemma 2.2. The middle columns is the direct sum of the fundamental short exact sequences

$$0 \longrightarrow \mathcal{O}(-F^\pm) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{F^\pm} \longrightarrow 0$$

together with the following exact sequences (cf. [Bea96]) for each $E_i \cap E_j = p \in Z$

$$0 \longrightarrow \mathcal{O}(-E_i - E_j) \longrightarrow \mathcal{O}(-E_i) \oplus \mathcal{O}(-E_j) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_p \longrightarrow 0$$

where we replace the first two sheaves with their quotient to obtain short sequences. The middle column is simply the exact sequence of modules

$$0 \longrightarrow \mathbb{Z} \xrightarrow{(1, \dots, 1)} \mathbb{Z}^{r+2} \longrightarrow \mathbb{Z}^{r+1} \longrightarrow 0$$

after tensoring by \mathcal{O}_X . The exactness of the sequences, together with the commutativity of both squares (easy to check), ensures the existence of an exact sequence on the left column. \square

We can now prove the first theorem stated in the introduction.

Proof of Theorem 1. It is direct from Lemmas 2.2 and 2.3 since $\text{im}(\alpha)$ in \mathcal{O}_X is isomorphic to \mathcal{Q}_α . \square

Proposition 2.4. *The following holds: $\mathcal{E}xt^1(\mathcal{Q}_\alpha, \mathcal{O}_X) \cong \mathcal{O}_Z$.*

Proof. By the definition of \mathcal{Q}_α , Lemma 2.2 and the long exact sequence for ext sheaves we have $\mathcal{E}xt^1(\mathcal{Q}_\alpha, \mathcal{O}_X) \cong \mathcal{E}xt^2(\mathcal{F}, \mathcal{O}_X)$. Since the functor $\mathcal{E}xt^i$ commutes with finite direct sums, it is enough to show that $\mathcal{E}xt^2(\mathcal{O}_{F^\pm}, \mathcal{O}_X) = 0$ and $\mathcal{E}xt^2(\mathcal{O}_{p_i}, \mathcal{O}_X) \cong \mathcal{O}_p$ for any $p \in Z$. Taking the long exact $\mathcal{E}xt$ -sequence of the exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_X(-F^\pm) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{F^\pm} \longrightarrow 0$$

and using $\mathcal{E}xt^i(\mathcal{O}_X, \mathcal{O}_X) = 0$ for any $i > 0$, by [Har77, Pro. III.6.3(b)], we get $\mathcal{E}xt^1(\mathcal{O}_X(-F^\pm), \mathcal{O}_X) \cong \mathcal{E}xt^2(\mathcal{O}_{F^\pm}, \mathcal{O}_X)$. By [Har77, Pro. III.6.7] we conclude that these sheaves are the zero sheaf, proving the first vanishing. To prove the second isomorphism observe that for each $p \in Z$ lying in the intersection $E_i \cap E_j$ we have the following exact sequence of sheaves [Bea96]

$$0 \longrightarrow \mathcal{O}_X(-E_i - E_j) \longrightarrow \mathcal{O}_X(-E_i) \oplus \mathcal{O}_X(-E_j) \xrightarrow{\varphi} \mathcal{O}_X \longrightarrow \mathcal{O}_p \longrightarrow 0.$$

Denoting by \mathcal{N} the quotient sheaf $\mathcal{O}_X(-E_i) \oplus \mathcal{O}_X(-E_j) / \mathcal{O}_X(-E_i - E_j)$ we deduce $\mathcal{E}xt^1(\mathcal{N}, \mathcal{O}_X) \cong \mathcal{E}xt^2(\mathcal{O}_p, \mathcal{O}_X)$ and the fact that $\mathcal{E}xt^1(\mathcal{N}, \mathcal{O}_X)$ is the cokernel of the map $\mathcal{O}_X(E_i) \oplus \mathcal{O}_X(E_j) \rightarrow \mathcal{O}_X(E_i + E_j)$ induced by φ taking tensor product with $\mathcal{O}_X(E_i + E_j)$. This proves the statement. \square

3. APPLICATIONS

Lemma 3.1. *Let $\varphi: \tilde{X} \rightarrow X$ be the blow-up of a smooth projective variety at a point $p \in X$. Then $h^1(\tilde{X}, T_{\tilde{X}}) \geq h^1(X, T_X)$.*

Proof. Since φ is a blow-up it follows that $R^i\varphi_*T_{\tilde{X}}$ vanishes for any $i > 0$. Thus the equality $h^i(\tilde{X}, T_{\tilde{X}}) = h^i(X, \varphi_*T_{\tilde{X}})$ holds for any i by [Har77, Exercise III.8.1] and we conclude by the following exact sequence of sheaves

$$0 \longrightarrow \varphi_*T_{\tilde{X}} \longrightarrow T_X \longrightarrow T_p \longrightarrow 0.$$

\square

Proof of Theorem 2. We begin by showing (1) \Rightarrow (2). Consider the good quotient map $\pi: X \rightarrow Y$. Assume first that the curve Y has positive genus. If π has only irreducible fibers, X is a ruled surface so by [Sei92, Theorem 4] we have $h^1(X, T_X) > 0$. This still holds if there are reducible fibers, by Lemma 3.1, because X would be a blow-up of one of such ruled surfaces. Thus, Y must necessarily be rational.

We show now that X contains no invariant rational curves C with $C^2 = -n \leq -2$. Suppose such a curve exists. From Lemma 2.2, after tensoring by $\mathcal{O}(K_X + C)$, we have an exact sequence

$$0 \rightarrow \pi^*(\Omega_{\mathbb{P}}^1) \otimes \mathcal{O}(E_S + K_X + C) \rightarrow \Omega_X(K_X + C) \rightarrow \text{im}(\alpha) \otimes \mathcal{O}(K_X + C) \rightarrow 0.$$

Let us compute some cohomology groups for these sheaves. Assume that $K_X + C$ is linearly equivalent to an effective divisor. From the genus formula we have

$$(K_X + C) \cdot C = 2g(C) - 2 = -2 < 0,$$

so by applying [ADHL13, Proposition V.1.1.2] we see that C must be in the base locus of $|K_X + C|$, meaning $K_X + C \sim C + E'$ for some effective divisor E' . This would imply that K_X is linearly equivalent to an effective divisor, a contradiction because X is rational and smooth. Thus $h^0(X, K_X + C) = 0$. Since $\text{im}(\alpha) \otimes \mathcal{O}(C + K)$ injects into $\mathcal{O}(C + K)$, then also

$$h^0(X, \text{im}(\alpha) \otimes \mathcal{O}(C + K)) = 0.$$

If F is a general fiber of π , the genus formula yields $F \cdot K_X = -2$. The product $F \cdot C$ equals at most 1 (where the equality holds if C is the source or sink curve), so

$$F \cdot (-2F + E_S + K_X + C) = F \cdot K_X + F \cdot C < 0.$$

Then $h^0(X, \pi^*(\Omega_{\mathbb{P}}^1) \otimes \mathcal{O}(E_S + K_X + C)) = h^0(X, -2F + E_S + K_X + C) = 0$. Going back to the exact sequence, we can now deduce that $h^0(X, \Omega_X(K_X + C)) = 0$, and due to Serre's duality we conclude $h^2(X, T_X(-C)) = 0$.

Consider now the exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_X(-C) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0.$$

Tensoring by T_X gives a new exact sequence

$$0 \longrightarrow T_X(-C) \longrightarrow T_X \longrightarrow T_X|_C \longrightarrow 0.$$

From the vanishing at H^2 shown above, there is a surjection $H^1(X, T_X) \rightarrow H^1(X, T_X|_C)$, so it suffices to show that $h^1(X, T_X|_C) \neq 0$ to prove the non-existence of this curve C , but this comes directly from the exact sequence

$$0 \longrightarrow T_C \longrightarrow T_X|_C \longrightarrow N_{C|X} \longrightarrow 0$$

and the fact that $h^1(X, T_C) = h^2(X, T_C) = 0$ and $h^1(X, N_{C|X}) = n - 1$.

We showed that any invariant rational curve of X has self-intersection ≥ -1 . Since the classes of these curves generate the Mori cone of X (see [ADHL13]) we conclude that $-K_X$ is ample and thus X is del Pezzo. Moreover by [Hug13, Proposition 5.9] del Pezzo \mathbb{K}^* -surfaces without elliptic fixed points are toric.

The proof of (2) \Rightarrow (1) is a consequence of [Ilt11, Corollary 2.8].

□

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